Math behind trajectories in SSMN

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Abstract

This document describes, how the trajectories are transformed into room-coordinates from a mathematical point of view and how to implement them. We assume, that we have just one instrument with one voice.

1 Introduction

Lets have a look on how to define a curve:

Definition 1. *A* ***parameterized curve*** *is a piecewise $C^\infty-$function defined on a closed and bounded interval I:*

$$c(t)\colon \mathbb{R} \supset I \to \mathbb{R}^n$$

In our application, t represents the tick given by MuseScore and the interval $I = [0, Length of the piece]$ represents the domain of the piece. Each trajectory t_i in the score defines the parameterized curve $\hat{f}_i$ on the interval $I_i = [a_i, b_i] \subset I$, where a_i represents the startpoint and b_i the endpoint of the scope and $d_i := b_i - a_i$ the duration. Assuming, that the intervals I_i are disjoint and $\bigcup_i I_i = I$, we can define the global parameterized curve as

$$f(t)\colon = \sum_i \mathbb{1}_{I_i} \cdot \hat{f}_i(t)$$

Trajectories don't care about their absolute position in the score. Therefore we define $f_i(t)\colon [0, d_i] \to \mathbb{R}^n, t \mapsto \hat{f}_i(t + a_i)$, which translates each trajectory ~~such that each starts~~ at time zero. This leads to:

$$f(t) = \sum_i \mathbb{1}_{I_i} \cdot f_i(t - a_i)$$

1.1 Dismantling of a trajectory

Theorem 1. *Trajectory $f_i(t)\colon [0, d_i] \to \mathbb{R}^n$, with $|f_i(t)| > 0,\ \forall t \in [0, d_i]$ can be divided into two functions $g_i(t)\colon [0, d_i] \to [0, 1]$ and $h_i(t)\colon [0, 1] \to \mathbb{R}^n$ ~~such~~ that:*

1. $f_i = g_i \circ h_i$
2. $f_i(0) = h_i(0)$ *and* $f_i(d_i) = h_i(1)$
3. $|h_i(t)| = const,\ \forall t \in [0, 1]$

Proof. This is a simple consequence from basic geometry lecture. □

The advantage of this separation is, that $g_i(t)$ can be interpreted as the function which defines the *acceleration* of the trajectory for a given time and $h_i(t)$ is just the *projection* that maps onto the trace of the trajectory. We can analyze those two classes of functions independently.

2 Acceleration functions

From the previous section, it follows that:

Definition 1. $g(t) : [0, d] \to [0, 1]$ *is called an* ***acceleration function*** *iff*

1. $g(0) = 0$
2. $g(d) = 1$

Theorem 1. $g(t)$ *is an* ***acceleration function*** $\implies \dot{g}(0) \geq 0$ *and* $\dot{g}(1) \geq 0$.

Proof. Trivial. □

2.1 Constant Acceleration

We want to construct an acceleration function $g(t) : [0, d] \to [0, 1]$with constant acceleration and given start speed. Our function must therefore fullfill:

1. $g(0) = 0$
2. $g(d) = 1$
3. Constant acceleration $\implies \ddot{g}(t) = const, \ \forall t \in [0, d]$
4. Fixed start speed $\implies \dot{g}(0) = v_0$

Let us start with requirement 3:

$$\begin{aligned} \ddot{g}(t) &= A \\ \iff \dot{g}(t) &= \int \ddot{g}(t) = At + B \\ \iff g(t) &= \int \dot{g}(t) = \frac{1}{2}At^2 + Bt + C \end{aligned}$$

We should respect 1:

$$0 \stackrel{!}{=} g(0) = C \implies C = 0$$

4 leads to:

$$v_0 \stackrel{!}{=} \dot{g}(0) = B \implies B = v_0$$

Using this information leads to:

$$g(t) = \frac{1}{2}At^2 + v_0 t$$

We can now determine the constant A using 2:

$$\begin{aligned} 1 \overset{!}{=} g(d) = \frac{1}{2}Ad^2 + v_0 d \\ \iff \frac{1}{2}Ad^2 = 1 - v_0 d \\ \iff A = 2\frac{1 - v_0 d}{d^2} \end{aligned}$$

We now have defined $g(t)$:

$$g(t) = \frac{1 - v_0 d}{d^2} t^2 + v_0 t$$

2.1.1 Remark

We still need to check, that $g(t) \in [0, 1]\ \forall t \in [0, d]$. Since $g(t)$ is a convex or concave function and we know that $\dot{g}(0) \geq 0$, it is sufficent to select our constants in such a way that $\dot{g}(d) \geq 0$.

$$\begin{aligned} 0 \leq \dot{g}(d) = Ad + v_0 = 2\frac{1 - v_0 d}{d^2} d + v_0 = 2\frac{1 - v_0 d}{d} + v_0 \\ \iff 0 \leq 2(1 - v_0 d) + v_0 d = 2 - 2v_0 d + v_0 d = 2 - v_0 d \\ \iff v_0 d \leq 2 \\ \iff v_0 \leq \frac{2}{d} \end{aligned}$$

We have a restriction on the start speed: Start speed must not exceed $\frac{2}{duration}$.